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Tensor - Matrix

$A^i \rightarrow$ contravarient tensor

$A_i \rightarrow$ covarient tensor

contravarient tensor:

Let a position of vector is reprent as (2m, 3m, 4m). If we change the unit along x direction from m to mm (smaller unit) the vector increases as (2000 mm, 3m, 4m), the tensor also increases. So, velocity, displacement, accelarations are contravarient tensor.

covarient tensor:

Let electro static field E at any point (4 v/m, 3 v/m, 6 v/m) now we change the unit along x direction from m to mm (smaller unit) so the vector also decrease as (0.004 v/mm, 3 v/m, 6 v/m).

As like gradient is also a covarient tensor.

Mixed tensor: containts both contravarient and covarient component.

ex: Moment of inertia tensor.

- If rank of a tensor is r and contains n component. the no of changes = r^n

Introduction to Tensor

- A second order tensor is a linear transformation which transform a vector (or vector space) to a another vector (vector space).

  eg. $Ax = b$, A is second order tensor

- Two direction and one magnitude. (2nd order tensor)

- Stress tensor $[\sigma] = \begin{bmatrix} \sigma_{xy} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{xx} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$ | longitudinal direction and areal direction.

- $I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(n \times n)}$

$\alpha = F/m$
$j = \sigma E$
} This laws valid in isotropic medium passessing high symmetry.

conductivity, $\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$

$$j_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z$$

Notation and conventions:

consider N-dimentional space $x^1, x^2, x^3, \ldots, x^N$ be any set of co-ordinates in this space.

The N-dimentional space under consideration will be denoted by V_N.

Let $\bar{x}^\alpha$ $(1 \le \alpha \le N)$ be another set of co-ordinate in the same space V_N.

$$x^i \equiv x^i (\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots, \bar{x}^N)$$

$1 \le i \le N$

$$\bar{x}^\alpha \equiv \bar{x}_\alpha (x^1, x^2, x^3, \ldots, x^N)$$

$$dx_j = \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}_\alpha \quad [i \text{ is free index} \\ - \alpha \text{ is dummy index}]$$

$$d\bar{x}_\alpha = \sum_{i=1}^{N} \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i \quad [\alpha \text{ is free index} \\ - i \text{ is dummy index}]$$

a dummy index can be replaced by any other index which does not appear in the same term.

- $a_i, b_i, c_i, d_i \quad 1 \le i \le N$

$$a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_N b_N$$

$$a_i b_j c_j = a_i b_k c_k = a_i b_l c_l$$

the dummy index j can not be replaced by i.

$$a_i b_i c_i \equiv a_1 b_1 c_1 + a_2 b_2 c_2 + \dots + a_N b_N c_N$$

- $$a_i b_i c_i d_i \equiv a_1 b_1 c_1 d_1 + \dots + a_N b_N c_N d_N$$

- $$a_i b_i c_j d_j \equiv \sum_{i=1}^{N} a_i b_i \sum_{j=1}^{N} c_j d_j$$

$$= (a_1 b_1 + a_2 b_2 + \dots + a_N b_N) \\ (c_1 d_1 + c_2 d_2 + \dots + c_N d_N)$$

So $a_i b_i c_j d_j \neq a_i b_i c_i d_i$

$$a_i b_j c_j d_j = a_i b_i c_k d_k = a_l b_l c_i d_i$$

i, j are dummy index.

Kronecker delta:

$$\frac{dx^i}{dx^j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

we define the Kronecker delta

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\frac{dx^i}{dx^j} = \delta^i_j$$

- $\frac{d\bar{x}^\alpha}{d\bar{x}^\beta} = \delta^\alpha_\beta$

x^i are function of $\bar{x}^\alpha$

- $\frac{dx^i}{dx^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^j}$

- $\frac{\partial \bar{x}^\alpha}{\partial x^k} \cdot \frac{\partial x^k}{\partial \bar{x}^\beta} = \delta^\alpha_\beta$.

contravariant Vector:

Tensors are define as the properties of transformation under coordinate transformations.

- A physical entity be characterized by the N functions A^i when expressd in the x^i coordinate system.

with respect to $\bar{x}^\alpha$ let, the same entity be characterized by the component $\bar{A}^\alpha$

$$\left.\begin{aligned} \bar{A}_\alpha &= \frac{\partial \bar{x}^\alpha}{\partial x^i} A^i \\ A^i &= \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \bar{A}^\alpha \end{aligned}\right\}$$

Co variant Vector

$$\bar{A}_\alpha = \frac{\partial x^i}{\partial \bar{x}_\alpha} A_i$$

Q. velocity and accelaration are contravarient and gradient of a scalar function is covariant vector

Tensors of second Rank:

A set of N^2 functions A^{ij} are said to be the components of a contravariant tensor of rank two if they transform

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \cdot \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \cdot A^{ij}$$

A set of N^2 functions A_{ij} are said to be the convariant tensor of the second rank if they transform according to

$$\bar{A}_{\alpha\beta} = \frac{\partial x^{i}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A_{ij}$$

Mixed Tensor of rank 2:

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \cdot \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A^{i}_{j}$$

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}$$

(aa) The elements of an arbitary matrix do not perform the second order rank matrix component.

General Definition

The above definitions can be easily generalized for a tensor of arbitrary rank a set of N^{p+q} functions $A^{i_1, i_2, \ldots i_p}_{j_1, j_2, \ldots j_q}$ are said to be the component of a tensor of contravariant rank p and covariant rank q (of total rank $p+q$) if they transform under coordinate system transformation according to

$$\bar{A}^{\alpha_1 \alpha_2 \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \ldots i_p}_{l_1 l_2 \ldots l_q}$$

$a_x = \frac{d^2x}{dt^2}$, $a_y = \frac{d^2y}{dt^2}$, $a_z = \frac{d^2z}{dt^2}$

$$\bar{a}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} a^i \qquad i = 1, 2, 3.$$

$x^1, x^2, x^3 \equiv x, y, z$

$a^1 \equiv a_x$, $a^2 \equiv a_y$, $a^3 \equiv a_z$.

r, θ, ϕ be the polar co-ordinate,

$$a_r = \frac{\partial r}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial r}{\partial y}\cdot\frac{d^2y}{dt^2} + \frac{\partial r}{\partial z}\cdot\frac{d^2z}{dt^2}$$

$$a^\theta = \frac{\partial \theta}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial \theta}{\partial y}\cdot\frac{d^2y}{dt^2} + \frac{\partial \theta}{\partial z}\frac{d^2z}{dt^2}$$

$$a^\phi = \frac{\partial \phi}{\partial x}\cdot\frac{d^2x}{dt^2} + \frac{\partial \phi}{\partial y}\cdot\frac{d^2y}{dt^2} + \frac{\partial \phi}{\partial z}\cdot\frac{d^2z}{dt^2}$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx = dr\sin\theta\cos\phi$

$\frac{\partial r}{\partial x} = \sin\theta\sin\phi$

$\frac{\partial r}{\partial y} =$

$\frac{\partial r}{\partial z} =$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$$

$$a_\theta = \ddot{\theta} - \sin\theta\cos\phi\,\dot{\phi}^2$$

$$a_\phi = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi}$$

Tensor & Matrix

26.08.2019

Transformation law

$$\bar{A}^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdot \frac{\partial \bar{x}^{i_2}}{\partial x^{m_2}} \cdot \ldots \cdot \frac{\partial x^{\beta_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{\beta_2}}{\partial \bar{x}^{j_2}} A^{\mu_1 \mu_2 \mu_3 \ldots \mu_n}_{\beta_1 \beta_2 \ldots \beta_n}$$

Rank-2n

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{a}} \frac{\partial \bar{x}^{\beta}}{\partial x^{b}} A^{ab}$$

Kroneker Delta:

$$\delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} = \frac{\partial x^i}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial \bar{x}^j}$$

$i = 1, \ldots, n$

#Q. $\delta^i_i = \delta^j_j = \frac{\partial x^1}{\partial x^1} \cdot \frac{\partial x^2}{\partial x^2} \cdot \ldots \cdot \frac{\partial x^n}{\partial x^n} + \ldots \ldots + \frac{\partial x^1}{\partial x^2} \cdot \frac{\partial x^2}{\partial x^2} \cdot \ldots \frac{\partial x^n}{\partial x^1} + \ldots \ldots$

$= 1 + 1 + \ldots$

$= n$ ✓

#Q. $\delta^i_j \delta^j_k = \delta^i_k$ $\quad j = 1, \ldots n$

or, $\frac{dx^i}{dx^j} \cdot \frac{dx^j}{dx^k} = \frac{dx^i}{dx^k} = \delta^i_k$

$\delta^i_j \delta^j_k = \delta^i_1 \delta^1_k + \delta^i_2 \delta^2_k + \ldots \ldots + \delta^i_n \delta^n_k$

$= \delta^1_k + \delta^2_k + \ldots + \delta^n_k$ ✓

$= \delta^i_k$

Q1 prove that δ^i_j is invariant under co-ordinate transformation system.

Q2. prove that δ^i_j has transformation property as a mixed tensor of rank 2.

Q3. $a_{\alpha\beta} x^\alpha x^\beta = 0$ for all value of $x^1, x^2, \ldots x^n$ Show that $a_{\mu\nu} + a_{\nu\mu} = 0$.

=

$$\frac{d}{dx^\mu}(a_{\alpha\beta} x^\alpha x^\beta) = 0$$

$$\text{or, } a_{\alpha\beta} x^\beta \frac{\partial x^\alpha}{\partial x^\mu} + a_{\alpha\beta} x^\alpha \frac{\partial x^\beta}{\partial x^\mu} = 0$$

$$\text{or, } a_{\alpha\beta} x^\beta \delta^\alpha_\mu + a_{\alpha\beta} x^\alpha \delta^\beta_\mu = 0$$

$$\text{or, } a_{\mu\beta} x^\beta + a_{\alpha\mu} x^\alpha = 0$$

$$\frac{d}{dx^\nu}(a_{\alpha\beta} x^\alpha x^\beta) = 0$$

$$\text{or, } a_{\alpha\beta} x^\alpha \frac{\partial x^\beta}{\partial x^\nu} + a_{\alpha\beta} x^\beta \frac{\partial x^\alpha}{\partial x^\nu} = 0$$

$$\text{or, } a_{\alpha\beta} x^\alpha \delta^\beta_\nu + a_{\alpha\beta} x^\beta \delta^\alpha_\nu = 0$$

$$\text{or, } a_{\alpha\nu} x^\alpha + a_{\nu\beta} x^\beta = 0$$

$$\frac{d}{dx^\nu}(a_{\mu\beta} x^\beta + a_{\alpha\mu} x^\alpha) = 0$$

$$\text{or, } a_{\mu\beta} \frac{\partial x^\beta}{\partial x^\nu} + a_{\alpha\mu} \frac{\partial x^\alpha}{\partial x^\nu} = 0$$

$$\text{or, } a_{\mu\nu} + a_{\nu\mu} = 0$$

Q1. The property of Kroneeker delta

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Suppose we consider a kroneeker delta, δ^i_j converted into $\delta^{i'}_{j'}$ under co-ordinate transformation,

$$\delta^{i'}_{j'} = \delta^k_l \frac{\partial x_i}{\partial \bar{x}_k} \cdot \frac{\partial \bar{x}_j}{\partial x_{j'}}$$

$$\delta^{i'}_{j'} = \frac{\partial x_{i'}}{\partial x_{j'}}$$

$$\therefore \delta^{i'}_{j'} = \delta^k_k \cdot \frac{\partial x_{i'}}{\partial x_{j'}}$$

$$= \frac{\partial x_{i'}}{\partial x_{j'}}$$

Kroneeker delta,

$$\delta^k_l = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

$x^i \equiv x^i(x^1, x^2, \ldots x^N)$

$x^k = \bar{x}^k(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^N)$

$\therefore$ Kroneker delta remain invariant under co-ordinate transformation.

Q2.

under co-ordinate transformation

$$x^i \equiv x^i(x^1, x^2, \ldots x^N)$$

$$x^k \equiv \bar{x}^k(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^N)$$

of Kroneker delta function $\delta^{i'}_{j'} = \delta^k_l \frac{\partial x_{i'}}{\partial \bar{x}_k} \cdot \frac{\partial \bar{x}_l}{\partial x_{j'}}$

this shown that 1 rank of kroneker delta is for contravarient and 1 rank for covarient tensor property. so kroneker delta have rank 2 mixed tensor cherectaristics.

Show that velocity of a fluid is a contravarient tensor of rank 1.

= proof that, $\bar{v}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} v^\beta$

Let us consider a particle at time t and position x^α.

$$\therefore \bar{v}^\alpha = \frac{d\bar{x}^\alpha}{dt}, \quad v^\beta = \frac{dx^\beta}{dt}$$

$$= \frac{\partial x^\beta}{\partial \bar{x}^\alpha} \cdot \frac{d\bar{x}^\alpha}{dt}$$

$$v^\beta = \frac{dx^\beta}{d\bar{x}^\alpha} \cdot \bar{v}^\alpha$$

$$\therefore \boxed{\bar{v}_\alpha = \frac{d\bar{x}^\alpha}{dx^\beta} \cdot v^\beta}$$

velocity is a contravarient tensor of rank 1.

Show that accelaration is contravarient tensor.

Show that gradient of a scalar field is covarient tensor.

If $A^{\alpha\beta\gamma} = A^{\beta\alpha\gamma}$, A is called Symmetric tensor because with the interchange of indices A remain same. (value)

If $A^{\alpha\beta\gamma} = -A^{\beta\alpha\gamma}$, A is called anti-symmetric matrix because with the interchange of indice A remain same with

Show that symmetric property of tensor are invarient.

= $A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$, show that $\bar{A}_{\alpha\beta\gamma} = \bar{A}_{\beta\alpha\gamma}$

$$\bar{A}_{\alpha\beta\gamma} = \frac{\partial x^a}{\partial \bar{x}_\alpha} \cdot \frac{\partial x^b}{\partial \bar{x}_\beta} \cdot \frac{\partial x^c}{\partial \bar{x}_\gamma} A_{abc}$$

$$\bar{A}_{\beta\alpha\gamma} = \frac{\partial x^a}{\partial \bar{x}_\beta} \cdot \frac{\partial x^b}{\partial \bar{x}_\alpha} \cdot \frac{\partial x^c}{\partial \bar{x}_\gamma} A_{abc}$$

As, $A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$, so α an β can be interchange.
Symmetric property remain invarient.

Show that every tensor expressed assum of symmetric and anti-symmetric tensor with interchange of same kind of indices.

= $A^{\alpha\beta} = B^{\alpha\beta} + C^{\alpha\beta}$

$$A^{\alpha\beta} = \frac{1}{2}\left[A^{\alpha\beta} - A^{\beta\alpha}\right] + \frac{1}{2}\left[A^{\alpha\beta} + A^{\beta\alpha}\right]$$

$$= \frac{1}{2}\left[D^{\alpha\beta}\right] + \frac{1}{2} C^{\alpha\beta}$$

$$\therefore C^{\alpha\beta} = \left[A^{\alpha\beta} + A^{\beta\alpha}\right]$$

Applying symmetric property,

$$C^{\beta\alpha} = A^{\beta\alpha} + A^{\alpha\beta}$$
$$= C^{\alpha\beta}$$

$$D^{\alpha\beta} = A^{\alpha\beta} - A^{\beta\alpha}$$
$$C^{\beta\alpha} = A^{\beta\alpha} - A^{\alpha\beta}$$
$$= -(A^{\alpha\beta} - A^{\beta\alpha}) = -D^{\alpha\beta}$$

| symmetric property.

So we can say that every tensor expressed

show that $\frac{\partial A_j}{\partial x_i}$ is not a tensor, Although A_j is a covarient tensor of rank 1.

Addition of tensor: The sum of two and more tensor of same rank and same type; is same type of covarient and ..... ; same type of contravarient.

Same rule is used in subtraction.

Levi-civita tensor:

symbol ϵ^{ijk} / ϵ_{ijk}, its a anti-symmetric tensor.

$$\epsilon^{ijk} = -\epsilon^{jik}$$

$$\epsilon_{ijk} = -\epsilon_{jik} = \epsilon_{jki} = -\epsilon_{ikj} = \epsilon_{kij} = -\epsilon_{kji}$$

For even no of changes $\epsilon^{ijk} = \epsilon^{kji}$

If repeated index $\epsilon_{iik} = 0$.

Thus it is clear that whenever two indices are equal the corresponding co-efficient is zero, thus also a tensor of rank 3 in 3 dimention among 6 components are non-vanishing all of them have value +1 or -1. Therefore the only non-vanishing components of i, j, k for which all the indices will be different they are equal to +1 and -1. According as (ijk) is an even or odd permutation of 123. Therefore we redefine,

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation} \\ -1 & \text{if } ijk \text{ is an odd permutation} \\ 0 & \text{otherwise. (when two indices repeated)} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$$

12312

32132

$\vec{D} = \vec{A} \times \vec{B}$, where $D_1 = A_2 B_3 - A_3 B_2$

$$= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2$$

$$= \varepsilon_{1ij} A_i B_j$$

$$D_2 = \varepsilon_{2ij} A_i B_j$$

$$D_3 = \varepsilon_{3ij} A_i B_j$$

$$\boxed{D_\mu = \varepsilon_{\mu ij} A_i B_j} \qquad \mu = 1, 2, 3$$

Represent $\vec{R} = \vec{c} \cdot (\vec{A} \times \vec{B})$

$$= \vec{c} \cdot \vec{D}$$

$$= c_\mu D_\mu$$

$$= c_\mu \varepsilon_{\mu ij} A_i B_j$$

$$= \varepsilon_{\mu ij} c_\mu A_i B_j$$

Represent vector triple product interms of Levi-civita

$\delta_{ik}\, \varepsilon_{ikm} = 0$

$\frac{\partial x_i}{\partial x_k} \cdot \varepsilon_{ikm}$

if $i = k$,

Show that accelaration is a contravarient tensor of rank 1.

== Let us consider a particle at velocity $\bar{v}^\alpha$ at time t.

$\therefore$ accelaration $f = \frac{d\bar{v}^\alpha}{dt}$

Let we transform from

$\bar{x}^\alpha (\bar{x}_1, \bar{x}_2, \ldots)$

to $x^\beta (x_1, x_2, \ldots)$

co-ordinate system.

w.r.t x^β co-ordinate system particle, has velocity v^β at time t.

$$f^\beta = \frac{dv^\beta}{dt}.$$

$$= \frac{dv^\beta}{d\bar{v}^\alpha} \frac{d\bar{v}^\alpha}{dt}$$

$$= \frac{dv^\beta}{d\bar{v}^\alpha} \cdot \bar{f}^\alpha$$

this is similar transformation as,

$$A^\beta = \frac{dx^\beta}{\partial \bar{x}^\alpha} \bar{A}^\alpha$$

$\therefore$ accelaration is a contravarient tensor of rank 1.

Show that gradient of a scalar field is a covarient tensor of rank 1.

== Let ϕ is a scalar field.

gradient of scalar field in $x^\alpha (x_1, x_2, \ldots)$ co-ordinate system,

$$A^\alpha = \frac{d\phi}{dx^\alpha}$$

gradient of this scalar field in $\bar{x}^\beta (\bar{x}_1, \bar{x}_2, \ldots$ $\ldots)$

$$\bar{A}^\beta = \frac{d\phi}{d\bar{x}^\beta} = \frac{dx^\alpha}{d\bar{x}^\beta} \frac{d\phi}{dx^\alpha} = \frac{dx^\alpha}{d\bar{x}^\beta} \cdot A^\alpha$$

this shows that gradient of a scalar field is a covarient tensor of rank 1.

Show that $\frac{\partial A_\lambda}{\partial x_\mu}$ is not a tensor, Although A_λ is a covarient tensor of rank 1.

= As A_λ is a covarient tensor its transformation to x_α co-ordinate from x_λ co-ordinate written as

$$A_\lambda = \frac{\partial x_\alpha}{\partial x_\lambda} A_\alpha$$

Now, $$\frac{\partial A_\lambda}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\left(\frac{\partial x_\alpha}{\partial x_\lambda} A_\alpha\right)$$

$$= \frac{\partial x_\alpha}{\partial x_\lambda}\frac{\partial A_\alpha}{\partial x_\mu} + A_\alpha \frac{\partial}{\partial x_\mu}\left(\frac{\partial x_\alpha}{\partial x_\lambda}\right)$$

$$= \frac{\partial x_\alpha}{\partial x_\lambda}\frac{\partial A_\alpha}{\partial x_\mu} + A_\alpha \frac{\partial}{\partial x_\lambda}\left(\frac{\partial x_\alpha}{\partial x_\lambda}\right)\cdot\frac{\partial x_\lambda}{\partial x_\mu}$$

$$\therefore \quad \frac{\partial A_\lambda}{\partial x_\mu} = \frac{\partial x_\alpha}{\partial x_\lambda}\frac{\partial A_\alpha}{\partial x_\mu} + A_\alpha \frac{\partial^2 x_\alpha}{\partial x_\lambda^2}\cdot\frac{\partial x_\lambda}{\partial x_\mu}$$

It shown that the above equation is not for a tensor transformation to one-co-ordinate to another co-ordinate.

So, $\frac{\partial A_\lambda}{\partial x_\mu}$ is not a tensor.

Represent vector triple product as terms of Levi-civita tensor.

$$= A \times B \times C$$

$$B \times C = i(B_2C_3 - B_3C_2) + j(C_1B_3 - C_3B_1) + K(B_1C_2 - B_2C_1)$$

$$A \times B \times C = \hat{i} A_2(B_1C_2 - B_2C_1) + \hat{j} A_3(B_2C_3 - B_3C_2) + \hat{K} A_1(C_1B_3 - C_3B_1)$$

$$= \epsilon_{mli} A_l (\epsilon_{ijk} B_j C_k)$$

$i = 1,2,3$
$l = 1,2,3$
$m = 1,2,3$

$l=2, j=1, K=2$
$i=3$
ϵ_{123}

Sum / subtraction of two tensors of same rank and type results in a third tensor of same rank and type.

$$A_{\mu\nu} (\mp) B_{\mu\nu} = C_{\mu\nu}$$

outer product between two tensor results a new tensor, with rank sum of two tensor

$$\boxed{A_{\mu\nu} \cdot B_{\mu\nu} = C_{abcd}}$$

$=$ Say $A^{\alpha\alpha}_{\beta} B^{\phi}_{c} = C^{\alpha\alpha\phi}_{\beta c}$ [rank = 5]

$$\bar{A}^{\alpha}_{\gamma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{a}} \cdot \frac{\partial \bar{x}^{\beta}}{\partial x^{b}} \cdot \frac{\partial x^{\delta}}{\partial \bar{x}^{\gamma}} A^{ab}_{e}$$

$$\bar{B}^{\phi}_{\delta} = \frac{\partial \bar{x}^{\phi}}{\partial x^{d}} \cdot \frac{\partial x^{e}}{\partial \bar{x}^{\delta}} B^{d}_{e}$$

$$\bar{A}^{\alpha\beta}_{\gamma} \cdot \bar{B}^{\rho}_{q} = \cdot \qquad A^{ab}_{c} \; B^{x}_{y}$$

$$\bar{C}^{\alpha\beta\rho}_{\gamma q} = \qquad C^{abx}_{cy}$$

contraction:

$$A^{\mu\nu\sigma}_{\lambda\rho}$$

$$\bar{A}^{\mu\nu\sigma}_{\lambda\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{a}} \cdot \frac{\partial \bar{x}^{\nu}}{\partial x^{b}} \cdot \frac{\partial \bar{x}^{\sigma}}{\partial x^{c}} \cdot \frac{\partial x^{d}}{\partial \bar{x}_{\lambda}} \cdot \frac{\partial x^{e}}{\partial \bar{x}\rho} \cdot A^{abc}_{de}$$

Say $\rho = \sigma$,

$$\bar{A}^{\mu\nu\sigma}_{\lambda\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{a}} \cdot \frac{\partial \bar{x}^{\nu}}{\partial x^{b}} \cdot \frac{\partial \bar{x}^{\sigma}}{\partial x^{c}} \cdot \frac{\partial x^{d}}{\partial \bar{x}_{\lambda}} \cdot \frac{\partial x^{e}}{\partial \bar{x}^{\sigma}} A^{abc}_{de}$$

$$= \frac{\partial \bar{x}^{\mu}}{\partial x^{a}} \cdot \frac{\partial \bar{x}^{\nu}}{\partial x^{b}} \cdot \delta^{e}_{c} \frac{\partial x^{d}}{\partial \bar{x}_{\lambda}}$$

$$\bar{B}^{\mu\nu}_{\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{a}} \cdot \frac{\partial \bar{x}^{\nu}}{\partial x^{b}} \cdot \frac{\partial x^{\lambda}}{\partial \bar{x}^{\delta}} \; B^{ab}_{\delta}$$

Inner product:

Show that inner product between two tensors of rank r_1 and r_2 result a new tensor of rank $(r_1 + r_2 - 2)$.

Show that inner product of two tensor of rank one is the tensor of rank 0.

A^i and B_j are componenent of tensors rank 1 then show that $C^i_j = A^i B_j$

Matric Tensor / Fundamental Tensor:

Let us consider two very closly place point x^i and $x^i + dx^i$ in a n-dimentional space. Distance Between this two point's ds.

$$ds^2 = g_{ij}\, dx^i dx^j$$ where the co-efficient g_{ij} may be function of x_i and $|g_{ij}| \neq 0$

$g_{ij}(x_i)$ = rimeen Space.

g_{ij} known as matric tensor / Fundamental tensor

Obtain Matric for a two Dimentional plane inferm of cartesian co-ordinate system:

Let us consider two point P, Q with co-ordinate (x,y) and $(x+dx, y+dy)$

$$ds^2 = dx^2 + dy^2 + 0 \cdot dx \cdot dy + 0 \cdot dx \cdot dy$$

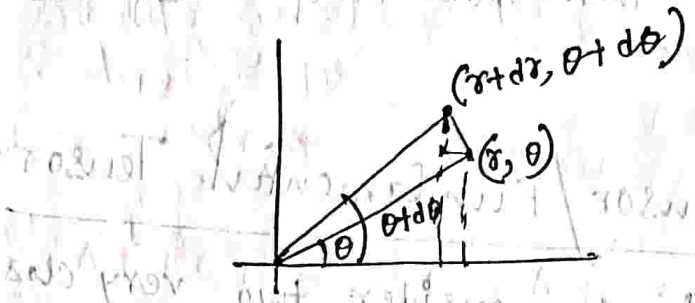
$g_{11} = 1$ $\quad g_{22} = 1$

$g_{12} = 0$ $\quad g_{21} = 0$

$$g_{ij} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

obtain the matric tensor for two-dimention plane. in polar co-ordinate.

two point (r, θ) $(r+dr, \theta+d\theta)$



$$arc = (r+dr)d\theta$$

$$\text{distance} = dr^2 + rd\theta^2$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

$$= dr^2 + r^2 d\theta^2 + 0.drd\theta + 0.d\theta dr$$

$$g_{11} = 1 \quad g_{22} = r^2$$

$$g_{12} = 0 \quad g_{21} = 0$$

$$g_{ij} = \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix}$$

Obtain Matric in spherical polar co-ordinte when the radius is a (constand).

consider a co-ordinate system (u, v, w) relation with certesian co-ordinate, $x = vw \quad y = uw \quad z = uv$

Q. sum and subtraction of two tensors of same rank and type results in a third tensor of same rank and type.

= we consider the matrix formed of two tensors $A_{\mu\nu}$ and $B_{\mu\nu}$ of their co-efficient.

$$A_{\mu\nu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B_{\mu\nu} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$A_{\mu\nu} + B_{\mu\nu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}+b_{11}) & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & b_{33}+a_{33} \end{pmatrix}$$

$$\left(\begin{matrix} \mu = 1, 3 \\ \nu = 1, 3 \end{matrix}\right)$$

So, $a_{\mu\nu} + b_{\mu\nu} = c_{\mu\nu}$

So we can write a new matrix of the co-efficient of tensor $C_{\mu\nu}$ such that.

$$A_{\mu\nu} + B_{\mu\nu} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = C_{\mu\nu}$$

$C_{\mu\nu}$ is same tensor of rank as $A_{\mu\nu}$ and $B_{\mu\nu}$

So, $C_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}$

Similarly we can find subtraction of two tensors of same type and rank form a tensor of same rank and type.

$$C_{\mu\nu} = A_{\mu\nu} - B_{\mu\nu}$$

we can write as,

$$\boxed{A_{\mu\nu} \pm B_{\mu\nu} = C_{\mu\nu}}$$

Subtraction or addition of tensor of same rank and same type forms a tensor of same rank and same type.

Q. Show that inner product of two tensors of rank 1 is the tensor of rank 0.

= we consider two tensors of rank 1,

X^a and Y_b.

the outer product of these tensor form a mixed tensor of rank 2.

$$X^a Y_b = Z^a_b$$

Now applying contraction process,

$\gamma = \beta$

$$\bar{X}^\gamma \bar{Y}_\beta = \frac{\partial \bar{x}_\gamma}{\partial x_a} \cdot \frac{\partial x_b}{\partial \bar{x}_\beta} Z^a_b$$

$$\frac{\partial \bar{x}_\gamma}{\partial x_a} \cdot \frac{\partial x_b}{\partial \bar{x}_\gamma} Z^a_b = \delta^b_a Z^a_b$$

when $a=b$, $\delta^b_a = 1$

$$\bar{Z}^\gamma_\gamma = Z^a_a$$

which is a scalar tensor.

Q. Show that inner product between two tensors of rank r_1 and r_2 results a tensor of rank (r_1+r_2-2).

= We consider two ..... tensors such that

$$A^{\alpha_1, \alpha_2, \ldots \alpha_{r_1}}$$

and $B_{\beta_1, \beta_2, \ldots, \beta_{r_2}}$

of rank r_1 and r_2 respectively. So the outer product of two tensor form a mixed tensor of rank r_1+r_2.

$$C^{\alpha_1, \alpha_2, \ldots, \alpha_{r_1}}_{\beta_1, \beta_2, \ldots \beta_{r_2}} = A^{\alpha_1, \alpha_2, \ldots \alpha_{r_1}} \cdot B_{\beta_1, \beta_2 \ldots \beta_{r_2}}$$

Let under co-ordinate transformation,

$$\bar{C}^{a_1, a_2, \ldots, a_{r_1}}_{b_1, b_2, \ldots, b_{r_2}} = \frac{\partial \bar{x}_{a_1}}{\partial x_{\alpha_1}} \cdot \frac{\partial \bar{x}_{a_2}}{\partial x_{\alpha_2}} \cdots \frac{\partial x_{\beta_1}}{\partial \bar{x}_{b_1}} \cdot \frac{\partial x_{\beta_2}}{\partial \bar{x}_{b_2}} \cdots C^{\alpha_1, \alpha_2 \ldots \alpha_{r_1}}_{\beta_1 \beta_2 \ldots \beta_{r_2}}$$

Let use contraction, $a_1 = b_1$

so, $\frac{\partial \bar{x}_{a_1}}{\partial x_{\alpha_1}} \cdot \frac{\partial x_{\beta_1}}{\partial \bar{x}_{a_1}} = \delta^{\beta_1}_{\alpha_1} = 1$ for $\beta_1 = \alpha_1$

$= 0$ for $\alpha_1 \neq \beta_1$

so, $\bar{C}^{a_2, \ldots a_{r_1}}_{b_2, \ldots b_{r_2}} = C^{\alpha_2, \alpha_3, \ldots \alpha_{r_1}}_{\beta_2, \beta_3, \ldots \beta_{r_2}}$

so, the inner product between two tensors of rank r_1 and r_2 result a tensor of rank (r_1+r_2-2).

Q. A^i and B_j are component of tensor of rank 1 then show that $C^i_j = A^i B_j$

= Under co-ordinate transformation

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha} A^\alpha$$

$$\bar{B}_j = \frac{\partial x^\beta}{\partial \bar{x}^j} B_\beta$$

so that $\bar{C}^i_j = \bar{A}^i \bar{B}_j = A^\alpha B_\beta \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^j}$.

C^i_j transformation as a mixed tensor of rank two.

Q. Obtain metric tensor in spherical polar co-ordinate system when the radius is a (constant)

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

here, $r = a$ (constant)

$$dx = a\left[-\sin\theta\sin\phi\, d\phi + \cos\phi\cos\theta\, d\theta\right]$$

$$dy = a\left[\sin\theta\cos\phi\, d\phi + \sin\phi\cos\theta\, d\theta\right]$$

$$dz = -a\sin\theta\, d\theta$$

$$\therefore ds^2 = dx^2 + dy^2 + dz^2$$

$$= a^2\sin^2\theta\sin^2\phi\, d\phi^2 + a^2\cos^2\phi\cos^2\theta\, d\theta^2 - 2a^2\sin\theta\sin\phi\cos\phi\cos\theta\, d\theta\, d\phi + a^2\sin^2\theta\cos^2\phi\, d\phi^2 + a^2\sin^2\phi\cos^2\theta\, d\theta^2 + 2a^2\sin\theta\sin\phi\cos\theta\cos\phi\, d\theta\, d\phi$$

$$+ a^2 \sin^2\theta \, d\theta^2$$

$$= a^2 \sin^2\theta \, d\varphi^2 (\sin^2\varphi + \cos^2\varphi) + a^2 \cos^2\theta \, d\theta^2 (\cos^2\varphi + \sin^2\varphi)$$

$$+ a^2 \sin^2\theta \, d\theta^2$$

$$= a^2 \sin^2\theta \, d\varphi^2 + a^2 d\theta^2 (\cos^2\theta + \sin^2\theta)$$

$$= a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$

$$g_{11} = a^2, \quad g_{22} = a^2 \sin^2\theta$$

$$g_{12} = 0, \quad g_{21} = 0$$

Metric tensor $g_{ij} = \begin{vmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{vmatrix}$

consider a co-ordinate system (u, v, w) relation with cartesian co-ordinate $x = vw$, $y = wu$ $z = vu$ find the metric tensor for the co-ordinate system given.

$$dx = v\,dw + w\,dv$$
$$dy = w\,du + u\,dw$$
$$dz = v\,du + u\,dv$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= v^2 dw^2 + w^2 dv^2 + 2vw\,du\,dv + w^2 du^2 + u^2 dw^2$$
$$+ 2wu\,du\,dw + v^2 du^2 + u^2 dv^2 + 2vu\,du\,dv$$

$$= (w^2 + v^2)\,du^2 + 2vw\,du\,dv$$
$$+ (w^2 + u^2)\,dv^2 + 2vw\,dv\,dw + (v^2 + u^2)\,dw^2$$

$$+ 2wu\,du\,dw$$

$g_{11} = w^2 + v^2$ $\quad g_{22} = w^2 + u^2$ $\quad g_{33} = v^2 + u^2$

$g_{12} = 2vw$ $\quad g_{23} = 2vu$ $\quad g_{31} = 2wu$

$g_{13} = 0$ $\quad g_{21} = 0$ $\quad g_{32} = 0$

$$g_{ij} = \begin{pmatrix} (w^2+v^2) & 2vu & 0 \\ 0 & w^2+u^2 & 2vw \\ 2wu & 0 & v^2+u^2 \end{pmatrix}$$

What is mean by invarient? Show that the contraction of outerproduct of the tensors A^p and $B^{\bullet}_{q}$ is an invarient.

= Suppose ϕ is a function of the co-ordinate x^k and let $\bar{\phi}$ denote the functional value under a transformation to a new set of co-ordinate $\bar{x}^k$ then ϕ is called a scalar / Invarient w.r.t co-ordinate transformation if $\phi = \bar{\phi}$.

= Here given two tensor,

$$\bar{A}^p = \frac{\partial x^p}{\partial x^a} A^a$$

$$\bar{B}^{\bullet}_{q} = \frac{\partial x^b}{\partial x^q} B^{\bullet}_{b}$$

$$\bar{A}^p \bar{B}^{\bullet}_{q} = \frac{\partial x^p}{\partial x^a} \cdot \frac{\partial x^q}{\partial x^b} A^a \cdot B^{\bullet}_{b}$$

Now applying contraction $p = q$

$$\bar{A}^p \bar{B}^{\bullet}_{q} = \frac{\partial x^p}{\partial x^q} \cdot \frac{\partial x^b}{\partial x^p} A^a \cdot B^{\bullet}_{b}$$

$$\bar{A}^p \bar{B}^{\bullet}_{p} = \delta^b_a A^a \cdot B^{\bullet}_{b} \qquad [\delta^b_a = 1, \text{ for } b = a]$$

$$= A^a \cdot B^{\bullet}_{a}$$

conjugate or Reciprocal tensor:

$$g^{pq} g_{pq} = \delta^p_r$$

where $g^{pq} = \frac{\text{co-factor of } g_{pq}}{g}$

$$g = |g_{pq}|$$

$A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i$

$$\frac{\partial (A^j)}{\partial x^k} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial x^k} + \frac{\partial^2 x^j}{\partial x^k \partial \bar{x}^i} \bar{A}^i$$

$$= \frac{\partial x^j}{\partial \bar{x}^i} \cdot \frac{\partial \bar{x}^m}{\partial x^k} \frac{\partial \bar{A}^i}{\partial \bar{x}^m} + \frac{\partial^2 x^j}{\partial x^k \partial \bar{x}^i} \bar{A}^i$$

Christoffel symbol:

first kind, $[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right)$

2nd kind $\left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = g^{sr} [pq, r]$ $r = 1, 2$

$$[pq, r] = [qp, r]$$

$$\left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = \left\{ \begin{matrix} s \\ qp \end{matrix} \right\}$$

so, symmetric

Q. Prove that $[pq, r] = g^{rs}\begin{Bmatrix} s \\ pq \end{Bmatrix}$

Q. prove that $\begin{Bmatrix} s \\ pq \end{Bmatrix} = g^{sr}\{pq, r\}$

Q. $\frac{\partial g_{pq}}{\partial x^m} = [pm, q] + [qm, p]$

Now, $ds^2 = r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$ for a particular system find the value.

$[12,2]$ $[22,1]$

$g_1 = r^2 \quad g_{2r} = r^2\sin^2\theta \quad g_{12} = g_{21} = 0$

$$g_{ij} = \begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}$$

$$|g| = r^4\sin^2\theta.$$

$$g^{11} = \frac{r^2\sin^2\theta}{r^4\sin^2\theta} = \frac{1}{r^2}$$

$$g^{22} = \frac{r^2}{r^4\sin^2\theta} = \frac{1}{r^2\sin^2\theta}$$

$$[22,1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right]$$

$$= \frac{1}{2}\left[- \frac{\partial(r^2\sin^2\theta)}{\partial\theta}\right]$$

$$= \frac{1}{2}\left[-2r^2\sin\theta\cos\theta\right]$$

$$= -r^2\sin\theta\cos\theta.$$

~~[12,2]~~

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = g^{11}\,[22,1] + g^{12}\,[22,2]$$

$$= \frac{1}{r^2}\left[\frac{1}{2}(-2r^2\sin\theta\cos\theta\right] + 0$$

$$= -\sin\theta\cos\theta$$

\# $\left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\}$

\# If $ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\, d\phi^2$

Now find the value of —

i) $[22,1]$

ii) $\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\}$

iii) $\left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\}$

iv) $\left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\}$

v) $\left\{ \begin{matrix} 23, 3 \end{matrix} \right\}$

\# If the co-ordi. components of tensors is zero in one co-ordinate system is zero for another co-ordinate system.

A^{ij} is a contravarient tensor, B_j covarient tensor $A^{ij}{}_{Bk}$ is a tensor of rank 3 but $A^{ij}B_j$ is a tensor of rank 1.

Q# Surface of Sphere is a two dimentional Riemann space find it metric tensor if its radius is fixed.

$\Rightarrow$ $ds^2 = dx^2 + dy^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dx = -r\sin\theta\, d\theta$

$dy = r\cos\theta\, d\theta$

$= r^2 d\theta^2$

$g_{11} = 0, \quad g_{12} = 0, \quad g_{21} = 0$

$g_{22} = r^2$

Metric tensor $\begin{pmatrix} 0 & 0 \\ 0 & r^2 \end{pmatrix}$.

8. prove that $[pq,r] = g_{rs}\begin{Bmatrix} s \\ pq \end{Bmatrix}$

we know that

$$\begin{Bmatrix} s \\ pq \end{Bmatrix} = g^{rs}[pq,r]$$

Multiplying both side by g_{ks}

$$g_{ks}\begin{Bmatrix} s \\ pq \end{Bmatrix} = g_{ks}g^{rs}[pq,r]$$

$$= \delta^r_k [pq,r]$$

$$g_{rs}\begin{Bmatrix} s \\ pq \end{Bmatrix} = [pq,r] \qquad \delta^r_k = 1 \text{ for } r=k$$

9. prove that $\begin{Bmatrix} s \\ pq \end{Bmatrix} = g^{sr}[pq,r]$

we know that

$$[pq,r] = g_{rs}\begin{Bmatrix} s \\ pq \end{Bmatrix}$$

Multiplying both side by g^{ks}

$$g^{ks}[pq,r] = g_{rs}g^{ks}\begin{Bmatrix} s \\ pq \end{Bmatrix}$$

$$g^{ks}[pq,r] = \delta^k_r \begin{Bmatrix} s \\ pq \end{Bmatrix}$$

$$g^{rs}[pq,r] = \begin{Bmatrix} s \\ pq \end{Bmatrix} \qquad \delta^k_r = 1 \text{ for } k=r.$$

prove that $\frac{\partial g_{pq}}{\partial x^m} = [pm,q] + [qm,p]$

$\Rightarrow [pm,q] + [qm,p]$

$= \frac{1}{2}\left(\frac{\partial g_{pq}}{\partial x^m} + \frac{\partial g_{mq}}{\partial x^p} - \frac{\partial g_{pm}}{\partial x^q}\right) + \frac{1}{2}\left(\frac{\partial g_{qp}}{\partial x^m} + \frac{\partial g_{mp}}{\partial x^q} - \frac{\partial g_{qm}}{\partial x^p}\right)$

$= \frac{1}{2}\frac{\partial g_{pq}}{\partial x^m} + \frac{1}{2}\frac{\partial g_{pq}}{\partial x^m}$

$= \frac{\partial g_{pq}}{\partial x^m}.$

$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$

$g_{11} = 1 \quad , \quad g_{22} = r^2 \quad ; \quad g_{33} = r^2\sin^2\theta.$

$= \quad x_1 = r, \; x_2 = \theta \; x_3 = \phi$

$g_{12} = g_{13} = g_{21} = g_{23} = g_{31} = g_{32} = 0.$

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin\theta \end{vmatrix} \quad \left[= r^4\sin^2\theta \right.$$

$$g^{11} = \frac{\text{co factor of } g_{11}}{g} = \frac{\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}}{r^4\sin^2\theta} = 1.$$

$$g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}}{r^4\sin^2\theta} = \frac{1}{r^2}$$

$$g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix}}{r^4\sin^2\theta} = \frac{1}{r^2\sin^2\theta}.$$

(2)

According to christoffel symbol of the first kind.

$$[ij,K] = \frac{1}{2}\left[\frac{\partial g_{iK}}{\partial x^i} + \frac{\partial g_{jK}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^K}\right]$$

(a) $[22,1] = \frac{1}{2}\left[\frac{\partial(0)}{\partial\theta} + \frac{\partial(0)}{\partial\theta} - \frac{\partial(r^2)}{\partial r}\right] = -r.$

(b) $[33,1] = \frac{1}{2}\left[\frac{\partial g_{31}}{\partial\phi} + \frac{\partial g_{31}}{\partial\phi} - \frac{\partial g_{33}}{\partial r}\right]$

$= -r\sin^2\theta.$

(c) $\begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = g^{31}[23,1] + g^{32}[23,2] + g^{33}[23,3]$

$= \frac{1}{r^2\sin^2\theta} \cdot r^2\sin\theta\cos\theta = \cot\theta.$

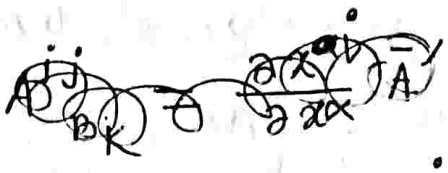
(d) $\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = g^{11}[33,1] + g^{12}[33,2] + g^{13}[33,3]$

$= 1\,(-r\sin^2\theta)$

$= -r\sin^2\theta.$

Q. A^{ij} is a contravarient tensor B_j covarient tensor $A^{ij}B_K$ is a tensor of rank 3 but $A^{ij}B_j$ is a tensor of rank 1.

$\Rightarrow$



$$A^{ij}B_K = \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\gamma}{\partial x^K} \bar{A}^{\alpha\beta} \bar{B}_\gamma$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^j}{\partial \bar{x}^\beta} \cdot \frac{\partial \bar{x}^\gamma}{\partial x^K} \cdot \bar{A}^{\alpha\beta} \bar{B}_\gamma$$

is a tensor of rank 3.

Now, $j = K$

$$A^{ij}B_j = \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^j}{\partial \bar{x}^\beta} \cdot \frac{\partial \bar{x}^\gamma}{\partial x^j} \bar{A}^{\alpha\beta} \bar{B}_\gamma$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \delta^\gamma_\beta \, \bar{A}^{\alpha\beta} \bar{B}_\gamma \qquad \delta^\gamma_\beta = 1 \text{ for } \gamma = \beta$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \bar{A}^{\alpha\beta} \bar{B}_\beta$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \bar{A}^{\alpha\beta} \bar{B}_\beta$$

has a rank 1.